DEVELOPMENT OF MAGNETOHYDRODYNAMIC BOUNDARY LAYERS

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Many authors have studied the problem of the development of a hydrodynamic boundary layer when a body is suddenly set in motion. The results obtained are presented most fully in the monographs of H. Schlichting [1] and L. G. Loitsyanskii [2]. In magnetohydrodynamics the development of the boundary layer over the surface of an infinite flat plate for uniform oncoming flow has been closely studied (for example [3, 4]). Below, the problem of the development of a plane magnetohydrodynamic boundary layer is considered in a different formulation. We shall suppose that the distributions of velocity U(x) and enthalpy $h_\infty(x)$ are given along the body contour for t = 0. At that moment the viscosity and thermal conductivity mechanisms are instantaneously "switched on". Viscous and thermal boundary layers begin to grow in a direction normal to the body. The medium in the boundary layer interacts with the magnetic field. This formulation corresponds to the development of a magnetohydrodynamic boundary layer on a body which is set in motion with a jerk, in the case where the rate of establishment of magnetohydrodynamic flow of the inviscid, thermally nonconducting fluid around the body is much less than the rate of development of the boundary layer. Then U(x) and $h_\infty(x)$ are found by solving the problem of stationary magnetohydrodynamic flow of an inviscid thermally nonconducting fluid around a body, or simply the hydrodynamic flow if the medium interacts with the field only in the boundary layer.

We shall consider a nonstationary magnetohydrodynamic boundary layer whose magnetic field vector lies in the plane of flow. We assume that the medium is incompressible (ρ = const), that the dynamic viscosity and thermal conductivity coefficients, as well as the Prandtl number P, are constant, and that the conductivity σ is isotropic. We further assume that together with the usual estimate for a boundary layer $R \gg 1$, the estimates $\Delta \gg \delta$, $R_m \ll 1$ hold (where δ and Δ are, respectively, the thickness of the boundary layer and the characteristic dimension of variation of the external magnetic field, which is independent of time) and R and R_m are the characteristic ordinary and magnetohydrodynamic Reynolds numbers, determined for the dimension δ. Then the boundary layer equations in the coordinate system attached to the body have the form [4, 5]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU_x' + \nu\frac{\partial^2 u}{\partial y^2} + \varepsilon(\sigma_\infty^\circ U - \sigma^\circ u) + e(\sigma_\infty^\circ - \sigma^\circ), \tag{2}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \nu^2 P^{-1}\frac{\partial^2 \theta}{\partial y^2} - 0.5\nu P^{-1}(1-P)\frac{\partial^2 u^2}{\partial y^2} + \sigma^\circ e(u + \varepsilon^{-1}e) \tag{3}$$

$$\left(\theta = h + \frac{u^2}{2},\ \sigma^\circ = \frac{\sigma}{\sigma_*},\ \varepsilon = \frac{\sigma_* H^2}{c^2\rho},\ e = \frac{\sigma_* EH}{c\rho}\right)$$

when the parameters at the edge of the boundary layer are independent of time.

Here the x and y coordinates are reckoned along the contour and in a direction normal to it, respectively, $U_x' = dU/dx$, h is the enthalpy, ν is the kinematic viscosity coefficient, θ is the stagnation enthalpy, σ_* and σ_∞° are the characteristic conductivity and dimensionless conductivity at the edge of the boundary layer, H = H(x) is the normal component of the applied magnetic field at points on the contour, E is the potential part of the electric field component in the z direction and independent of coordinates, and c is the velocity of light in vacuum. The functions $\varepsilon(x)$ and e are determined by the geometry of the applied electromagnetic field and the conditions of current flow. We note that in view of the inequality $(4\pi\sigma\nu/c^2) \ll 1$, the rotational component of the electric field in the boundary layer arising as a result of the time independence of the induced magnetic field components is negligibly small. In the general case the quantity E may depend on time; however, when currents flow freely in the z direction, then E = 0 if the magnetic field is fixed relative to the body and $E = -c^{-1}H_*V$ if the body moves with a velocity V = const in a uniform field H normal to the direction of motion.

For the solution of system (1)–(3) the boundary and initial conditions are

$$\begin{gathered} u = U(x), \quad \theta = \theta_\infty(x) \quad \text{for } 0 < y < \infty,\ t = 0;\ y = \infty,\ t \geqslant 0, \\ u = 0,\ v = 0,\ \theta = h_w(x) \quad \text{for } y = 0,\ t \geqslant 0, \\ (\theta_\infty(x) = h_\infty(x) + 0.5U^2, \quad (d\theta_\infty/dx) = e\sigma_\infty^\circ(U + e\varepsilon^{-1})). \end{gathered} \tag{4}$$

If the conductivity is uniform $\sigma^\circ = 1$, then equations (1) and (2) may be solved independently of equation (3). If $\sigma = \sigma(h)$, then the system (1)–(3) must be solved jointly.

We shall seek a solution of system (1)–(3) with boundary conditions (4) in the form

$$\begin{gathered} \psi = 2\sqrt{\nu t}\,[f_0(x,\eta) + tf_1(x,\eta) + t^2 f_2(x,\eta) + \ldots], \\ u = \frac{\partial\psi}{\partial y} = \frac{\partial f_0}{\partial\eta} + t\frac{\partial f_1}{\partial\eta} + t^2\frac{\partial f_2}{\partial\eta} + \ldots \qquad \left(\eta = \frac{y}{2\sqrt{\nu t}}\right), \\ v = -\frac{\partial\psi}{\partial x} = -2\sqrt{\nu t}\left[\frac{\partial f_0}{\partial x} + t\frac{\partial f_1}{\partial x} + t^2\frac{\partial f_2}{\partial x} + \ldots\right], \\ \theta = \theta_0(x,\eta) + t\theta_1(x,\eta) + t^2\theta_2(x,\eta) + \ldots \end{gathered} \tag{5}$$

The electrical conductivity is given by the series

$$\sigma^\circ = \sigma_0 + t\sigma_1 + t^2\sigma_2 + \ldots$$

whose coefficients are found with the help of the expansion for the enthalpy; σ_0 is calculated from the enthalpy $h_0 = \theta_0 - 0.5(\partial f_0/\partial\eta)^2$.

Setting (5) in system (1)–(3) and equating coefficients of like powers of t, we obtain the equations

$$\frac{\partial^3 f_0}{\partial\eta^3} + 2\eta\frac{\partial^2 f_0}{\partial\eta^2} = 0,$$

$$\left(\frac{\partial f_0}{\partial\eta}\right)_{\eta=0} = 0, \quad \left(\frac{\partial f_0}{\partial\eta}\right)_{\eta=\infty} = U, \quad \left(\frac{\partial f_0}{\partial x}\right)_{\eta=0} = 0 \tag{6}$$

$$\frac{\partial^3 f_1}{\partial\eta^3} + 2\eta\frac{\partial^2 f_1}{\partial\eta^2} - 4\frac{\partial f_1}{\partial\eta} = 2\Pi,$$

$$\left(\frac{\partial f_1}{\partial\eta}\right)_{\eta=0} = 0, \quad \left(\frac{\partial f_1}{\partial\eta}\right)_{\eta=\infty} = 0, \quad \left(\frac{\partial f_1}{\partial x}\right)_{\eta=0} = 0,$$

$$\Pi = \Pi(x, \eta) = 2\left(\frac{\partial f_0}{\partial\eta}\frac{\partial^2 f_0}{\partial\eta\,\partial x} - UU_x' - \frac{\partial f_0}{\partial x}\frac{\partial^2 f_0}{\partial\eta^2}\right) +$$

$$+ 2\varepsilon\left(\sigma_0\frac{\partial f_0}{\partial\eta} - \sigma_\infty^\circ U\right) + 2e(\sigma_0 - \sigma_\infty^\circ), \qquad (7)$$

$$\frac{\partial^2\theta_0}{\partial\lambda^2} + 2\lambda\frac{\partial\theta_0}{\partial\lambda} = \frac{0.5(1-P)}{P}\frac{\partial^2}{\partial\eta^2}\left[\left(\frac{\partial f_0}{\partial\eta}\right)^2\right],$$

$$\theta_0(x, 0) = h_w, \quad \theta_0(x, \infty) = \theta_\infty, \qquad (8)$$

$$\frac{\partial^2\theta_1}{\partial\lambda^2} + 2\lambda\frac{\partial\theta_1}{\partial\lambda} - 4\theta_1 = 2Q, \quad \theta_1(x, 0) = 0,$$

$$\theta_1(x, \infty) = 0 \quad (\lambda = \eta\sqrt{P}), \qquad (9)$$

$$Q(x, \lambda) = 2\left(\frac{\partial f_0}{\partial\eta}\frac{\partial\theta_0}{\partial x} - \frac{\partial f_0}{\partial x}\frac{\partial\theta_0}{\partial\eta}\right) +$$

$$+ \frac{0.5(1-P)}{P}\frac{\partial^2}{\partial\eta^2}\left(\frac{\partial f_0}{\partial\eta}\frac{\partial f_1}{\partial\eta}\right) - 2e\sigma_0\left(\frac{\partial f_0}{\partial\eta} + \frac{e}{\varepsilon}\right).$$

Equation (6) may be regarded as an ordinary linear homogeneous differential equation, in the function $\partial f_0/\partial\eta$, of the second order with respect to η, where x appears as a parameter. To solve it we employ the first two boundary conditions. The solution has the form

$$\frac{\partial f_0}{\partial\eta} = U(x)\operatorname{erf}\eta \qquad \left(\operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^\eta \exp(-\eta^2)\,d\eta\right). \quad (10)$$

Using the last boundary condition, we find

$$f_0 = U\int_0^\eta \operatorname{erf}\eta\,d\eta = U\{\eta\operatorname{erf}\eta + \pi^{-1/2}[\exp(-\eta^2) - 1]\}. \quad (11)$$

Equation (8) may be similarly integrated:

$$\theta_0 = h_w + 0.5\sqrt{\pi}A\operatorname{erf}\lambda + 0.5(1-P)r(\lambda),$$

$$A = 2\pi^{-1/2}[\theta_\infty - h_w - 0.5g\sqrt{\pi}(1-P)], \qquad (12)$$

$$g = \int_0^\infty G(\lambda)\left(\frac{\partial f_0}{\partial\eta}\right)^2 d\lambda,$$

$$r(\lambda) = \int_0^\lambda \exp(-\lambda^2)\left\{\int_0^\lambda\left\{\frac{\partial^2}{\partial\lambda^2}\left[\left(\frac{\partial f_0}{\partial\eta}\right)^2\right]\right\}\exp\lambda^2 d\lambda\right\}d\lambda, \qquad (13)$$

$$G(\lambda) = (1 + 2\lambda^2)\operatorname{Erf}\lambda\exp\lambda^2 - 2\pi^{-1/2}\lambda,$$

$$(G \geqslant 0, \ \operatorname{Erf}\lambda = 1 - \operatorname{erf}\lambda).$$

Taking (11), (12) into account, we reduce equation (7) to an ordinary linear differential equation in the function $\partial f_1/\partial\eta$, of the second order with respect to η, which is solved with the help of the first two boundary conditions. The last condition allows us to determine the function $f_1(x, \eta)$. We may subsequently integrate equation (9) as an ordinary differential equation in θ_1.

It was shown in [4] that the solution of equation

$$\Phi'' + 2\eta\Phi' - 4\Phi = 2\varphi(\eta), \quad \Phi(0) = 0, \quad \Phi(\infty) = 0 \qquad (14)$$

is the function $L(\eta) = L[\eta, \varphi(\eta)]$, which has the following properties:

$$\Phi = L[\eta, \varphi(\eta)], \qquad \Phi'(0) = -2\int_0^\infty G(\eta)\varphi(\eta)\,d\eta,$$

$$L[\eta, \varphi] \leqslant 0 \quad \text{if} \quad \varphi \geqslant 0 \quad (0 < \eta < \infty), \qquad (15)$$

$$L[\eta, k_1\varphi_1 + k_2\varphi_2] = k_1 L[\eta, \varphi_1] + k_2 L[\eta, \varphi_2] \quad (k_1, k_2 = \text{const}).$$

The expression for the function L and the graph of the function G determined from (13) are given in [4]. On the basis of (14), (15)

$$\frac{\partial f_1}{\partial\eta} = L[\eta, \Pi], \qquad \left(\frac{\partial^2 f_1}{\partial\eta^2}\right)_{\eta=0} = UU_x'N + UC, \quad (16)$$

$$N = -4\int_0^\infty G(\eta)\,\varkappa(\eta)\,d\eta,$$

$$C = 4\int_0^\infty G(\eta)\,[\varepsilon\sigma_\infty^\circ + eU^{-1}(\sigma_\infty^\circ - \sigma_0) - \varepsilon\sigma_0\operatorname{erf}\eta]\,d\eta,$$

$$\varkappa(\eta) = (\operatorname{erf}\eta)^2 - \left(\int_0^\eta \operatorname{erf}\eta\,d\eta\right)(\operatorname{erf}\eta)' - 1,$$

$$\theta_1 = L[\lambda, Q], \qquad \left(\frac{\partial\theta_1}{\partial\lambda}\right)_{\lambda=0} = s_1 + s_2 + s_3, \qquad (17)$$

$$s_1 = -4\int_0^\infty\left(\frac{\partial f_0}{\partial\eta}\frac{\partial\theta_0}{\partial x} - \frac{\partial f_0}{\partial x}\frac{\partial\theta_0}{\partial\eta}\right)G(\lambda)\,d\lambda,$$

$$s_2 = 4e\int_0^\infty G\sigma_0(\operatorname{erf}\eta + e\varepsilon^{-1})\,d\lambda,$$

$$s_3 = (P-1)\int_0^\infty \frac{\partial^2 G}{\partial\lambda^2}\frac{\partial f_0}{\partial\eta}\frac{\partial f_1}{\partial\eta}\,d\lambda.$$

The equations for approximations of higher order may also be reduced to the integration of ordinary differential equations, which may be carried out by quadratures [6].

We shall confine ourselves to a study of the zero-th and first approximations. The surface friction coefficient

$$c_f = U^{-1}\sqrt{\nu/t}\left\{\frac{2}{\sqrt{\pi}} + t\,[U_x'N + C] + \ldots\right\}. \quad (18)$$

The quantity N was calculated by Blasius [1]:

$$N = \frac{2}{\sqrt{\pi}}\left(1 + \frac{4}{3\pi}\right).$$

We shall consider the case of constant electrical conductivity $\sigma_\infty^\circ = \sigma_0 = 1$. Then

$$C = 4\varepsilon\int_0^\infty G(\eta)\operatorname{Erf}\eta\,d\eta = \frac{2\varepsilon}{\sqrt{\pi}},$$

$$c_f = \frac{2}{\sqrt{\pi}}\left(\frac{\nu}{t}\right)^{1/2}U^{-1}\left\{1 + t\left[\varepsilon + U_x'\left(1 + \frac{4}{3\pi}\right)\right] + \ldots\right\}. \qquad (19)$$

If boundary layer separation occurs, then the moment of separation, determined by taking only the two first approximations into account, equals

$$t^* = \left[-\varepsilon - U_x'\left(1 + \frac{4}{3\pi}\right)\right]^{-1}.$$

Since $\varepsilon \geqslant 0$, for separation to occur there must be points on the contour where $U_x' < 0$. Expressing U_x' in terms of the pressure gradient $p_x' = dp_\infty/dx$, we find

$$t^* = \left\{\frac{p_x'}{\rho U}\left(1 + \frac{4}{3\pi}\right) + \frac{e}{U}\left(1 + \frac{4}{3\pi}\right) - \frac{4\varepsilon}{3\pi}\right\}^{-1}. \qquad (20)$$

If the quantity in the braces is negative, then boundary layer separation will not occur. Since $\varepsilon \geqslant 0$, for an identical pressure distribution over the body the magnetohydrodynamic interaction determined by the force $c^{-1}\sigma\mathbf{v} \times \mathbf{H}$, always favors the earlier separation of the boundary layer. The electromagnetic interaction on the flow determined by the force $c^{-1}\mathbf{E} \times \mathbf{H}$, favors earlier separation if $e > 0$, and later separation if $e < 0$. These conclusions agree with the results of

[7] where the influence of the magnetic field on boundary layer separation was investigated for a stationary flow.

If the body moves with a velocity V in an external uniform field $\mathbf{H}_*$ and $\mathbf{V} \perp \mathbf{H}_*$, then

$$eU^{-1} = -\varepsilon (V / U)(H_* / H).$$

In the case of an infinite flat plate

$$p_x' = 0, \qquad eU^{-1} = -\varepsilon$$

and in accordance with (20), boundary layer separation will not occur.

We shall examine the case when the electric field is zero: e = 0. In this case the magnetic field is fixed relative to the body. We shall suppose that the flow outside the boundary layer does not interact with the field ($\sigma_\infty^c = 0$).

For P = 1 we find from (12), taking (10) into account,

$$h_0 = h_w + (h_\infty - h_w + 0.5U^2) \operatorname{erf} \eta - 0.5U^2 (\operatorname{erf} \eta)^2. \quad (21)$$

Let $\sigma(h_\infty) = 0$, and for $h > h_\infty$ let the electrical conductivity depend on the enthalpy according to a power law. Then, if $h_w \approx h_\infty$ (the electrical conductivity is enhanced due to kinetic energy dissipation),

$$\sigma_0 = \left(\frac{h_0}{h_*}\right)^n = \left(\frac{h_\infty}{h_*}\right)^n \left(\frac{h_0}{h_\infty}\right)^n \approx \alpha^n a_1^n(\eta),$$
$$\alpha = 0.5h_*^{-1}U^2, \qquad a_1(\eta) = \operatorname{Erf} \eta \operatorname{erf} \eta. \quad (22)$$

Here h* is the enthalpy from which the characteristic conductivity σ_* is determined.

If the velocities are small $h_w \gg h_\infty$ (gas hearing occurs as a result of heat transfer from the wall), then

$$\sigma_0 \approx \left(\frac{h_w}{h_*}\right)^n (\operatorname{Erf} \eta)^n. \quad (23)$$

Finally, if the surface is thermally isolated and P = 1, then we find from (8) that $\theta_0 = \text{const}$ and

$$\sigma_0 = \alpha^n a_2^n(\eta) \qquad (a_2 = 1 - (\operatorname{erf} \eta)^2). \quad (24)$$

Where (22) and (24) hold, the quantity C and the moment of separation (if it comes about) are equal:

$$C_k = -4\varepsilon\alpha^n \pi^{-1/2} i_k, \qquad t^* = \left[-U_x'\left(1 + \frac{4}{3\pi}\right) + 2\varepsilon\alpha^n i_k\right] \qquad (k = 1, 2),$$
$$i_1 = \sqrt{\pi} \int_0^\infty G(\eta) \operatorname{erf} \eta a_1^n d\eta, \qquad i_2 = \sqrt{\pi} \int_0^\infty G(\eta) \operatorname{erf} \eta a_2^n d\eta.$$

Values of the integrals i_1 and i_2 are given in [4]. When (23) holds and n = 1

$$C = -\frac{2h_w \varepsilon (4 - \pi)}{h_* \pi \sqrt{\pi}}, \qquad t^* = \left\{-U_x'\left(1 + \frac{4}{3\pi}\right) + \frac{h_w \varepsilon (4 - \pi)}{h_* \pi}\right\}.$$

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